Quadratic integral equations through Volterra-Stieltjes quadratic integral equation reformulation

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Abstract:

The study explores the existence of one and only one one solution for delay quadratic integral equation of the Volterra-Stieltjes type. Special cases include the delay quadratic integral equation and the Chandrasekhar integral equation

Keywords: delay quadratic integral equation, continuous solution, continuous dependence, Volterra-Stieltjes type

1. Introduction

Quadratic integral equations are prevalent in numerous real-world applictions. For instance, problems in radioactive transfer, neutron transport and the kinetic theory of gases often lead to such equations (see references[1-5]).

In particular, the existence of solutions for integral equations of the Volterra-Stieltjes type has been thoroughly studied by J. banas`([6-12]).

Consider the delay quadratic integral equation of Volterra-Stieltjes type

$$x(t) = p(t) + \int_0^{\varphi(t)} g_1\Big(t, s, x\big(\omega(s)\big)\Big)\, d_s k_1(t,s) \int_0^1 g_2\Big(t, s, x\big(\omega(s)\big)\Big)\, d_s k_2(t,s), \qquad t \in I = [0,1]. \quad (1)$$

This work focuses on examining the existence of solutions to equation (1) within the class of continuous functions. Furthermore, the dependence of the unique solution on the functions $k_1,\ k_2$ and φ will be analyzed.

As an application, the delay Volterra quadratic integral equations of Chandrasekhar type [1]

$$x(t) = a(t) + \int_0^{\varphi(t)} \frac{t}{t+s} k_1(t,s)|x(s)|ds \,.\, \int_0^1 \frac{t}{t+s} k_2(t,s)|x(s)|ds, \qquad (2)$$

will be given as example.

2 Existence of at least one solution

Now, equation (1) will be investigated under the assumptions

(i) $\varphi(t): [0,1] \to [0,1]$ is continuous and increasing such that $\varphi(t) \leq t$.

(ii) ω : is continuous.

(iii) $a \in C\,[0,1]$.

(iv) $g_i: [0,1] \times [0,1] \times R \to R$ are continuous and there exist the functions m_i and two positive constants b_i such that

$$|g_i(t,s,x)| \leq m_i(t,s) + b_i|x|$$

Where $m_i: [0,1] \times [0,1] \to R$, is continuous and

$$M = \sup_t\{m_i(t,s): t,s \in [0,1], i = 1,2\}.$$

Moreover, we put $b = \max_i\{b_i,\ i = 1,2\}$.

(v) (1) The function k_1 is continuous on Δ, where

$$\Delta = \{(t,s): 0 \leq s \leq t \leq 1\}.$$

(2) The function $k_2: [0,1] \times R \to R$ is continuous with

$$\mu = max\{sup\{|k_2(t,1)|: t \in [0,1]\}, \sup\{|k_2(t,0)|: t \in [0,1]\}\}.$$

(vi) (1) For each $\epsilon > 0$ there exists $\delta > 0$ for all $t_1, t_2 \in I$ such that $t_1 < t_2$ and

$t_2 - t_1 < \delta$ the following inequality holds:

$$\bigvee_0^{t_1}[k_1(t_2,s) - k_2(t_1,s)] \leq \epsilon.$$

(2) For all $t_1, t_2 \in I$ such that $t_1 < t_2$ the function $s \to k_2(t_2,s) - k_2(t_1,s)$ is nondecreasing on $[0,1]$.

(vii) (1) $k_1(t,0) = 0$ for any $t \in [0,1]$.

(2) $k_2(0,s)=0$ for any $s\in[0,1]$.

(viii) The function $s\to g_1(t,s)$ is of bounded variation on $[0,t]$ for each fixed $t\in I$.

(ix) $2(M+br)W\mu b<1$

(x) There exists a positive root r of the algebraic equation

$$\left(p+M^2W\mu\right)+b^2r^2W\mu+(MbW\mu-1)r=0.$$

Remark 1. The function $z\to \bigvee_{z=0}^{s} g_1(t,s)$ is continuous on $[0,t]$ for fixed $t\in I$ [7].

Lemma 1. For an arbitrary fixed $0<t_2<I$ and for any $\varepsilon>0$, there exists $\delta>0$ such that if $t_1\in I$, $t_1<t_2$ and $t_2-t_1<\delta$ then [7].

$$\bigvee_{s=t_1}^{t_2} k_1(t_2,s)\le\delta.$$

Lemma 2. [7] The function $t\to \bigvee_{s=0}^{t} g_1(t,s)$ is continuous on I. Then there exists a finite positive constant W such that

$$W=sup\left\{\bigvee_{s=0}^{t} k_1(t,s): t\in I\right\}.$$

Remark 2. The function $s\to k_2(t,s)$ is nondecreasing on the interval $[0,1]$. In fact for $s_1,s_2\in[0,1]$, with $s_1<s_2$, from assumptions (vi) and (vii), we obtain

$$k_2(t,s_2)-k_2(t,s_1)=[k_2(t,s_2)-k_2(0,s_2)]-[k_2(t,s_1)-k_2(0,s_1)]\ge 0.$$

Lemma 3. [7] Assume that the function g_2 satisfies assumption (vii). Then for arbitrary $s_1,s_2\in I$, such that $s_1<s_2$, the function $t\to k_2(t,s_2)-k_2(t,s_1)$ is nondecreasing on the interval I.
In fact, take for $t_1,t_2\in[0,1]$, with $t_1<t_2$. Then, by assumption (vii), we get

$$[k_2(t_2,s_2)-k_2(t_2,s_1)]-[k_2(t_1,s_2)-k_2(t_1,s_1)]=$$

$$[k_2(t_2,s_2)-k_2(t_1,s_2)]-[k_2(t_2,s_1)-k_2(t_1,s_1)\ge 0.$$

For the existence of at least one solution of the quadratic integral equation (1), we have the following theorem.

Theorem 1. Let assumptions (i) – (viii) be satisfied, then the functional integral equation (1) has at least one continuous solution $x\in C[0,1]$.

Proof. Define the operator

$$Ax(t) = p(t) + \int_0^{\varphi(t)} g_1\left(t, s, x(\omega(s))\right) d_s k_1(t,s) \int_0^1 g_2\left(t, s, x(\omega(s))\right) d_s k_2(t,s).$$

Define the Q **by** $Q = \{x \in C[0,1]: |x| \le r\}$, where r is a positive solution of the algebraic equation $(a + (m + br)\mu W = r)$.

$$|Ax(t)| = \left|p(t) + \int_0^{\varphi(t)} g_1\left(t, s, x(\omega(s))\right) d_s k_1(t,s) \int_0^1 g_2\left(t, s, x(\omega(s))\right) d_s k_2(t,s)\right|$$

$$\le p + \int_0^{\varphi(t)} \left|g_1\left(t, s, x(\omega(s))\right)\right| d_s k_1(t,s) \,.\, \int_0^1 \left|g_2\left(t, s, x(\omega(s))\right)\right| d_s k_2(t,s)$$

$$\le p + \int_0^{\varphi(t)} (m_1(t,s) + b_1|x|) d_s(\vee_{z=0}^{s} k_1(t,z)) \quad \int_0^1 (m_2(t,s) + b_2|x|) d_s(\vee_{z=0}^{s} k_2(t,z))$$

$$\le p + (M + b\|x\|) \int_0^{\varphi(t)} d_s k_1(t,s). (M + b\|x\|) \int_0^1 d_s k_2(t,s)$$

$$\le p + (M + br) \sup_{t \in I} \vee_{s=0}^{s} k_1(t,s). (M + br)(k_1(t,1) - k_2(t,0))$$

$$\le p + (M + br)^2 W\mu = r.$$

Hence, $Ax \in Q$ which proves that the operator F maps Q into itself and the class of functions $\{Ax\}$ is uniformly bounded in Q.
Let $x \in Q$ and define

$$\vartheta(\delta) = \sup_{x \in Q_r} \left\{\left|g_i\left(t_2, s, x(\omega(s))\right) - g_i\left(t_1, s, x(\omega(s))\right)\right| : t_1, t_2 \in [0,1],\ t_1 < t_2\,,\right.$$

$$\left. |t_2 - t_1| < \delta, s \in I, i = 1, 2\right\},$$

then from the uniform continuity the function $g_i: [0,1] \times [0,1] \times Q \to R$ and assumption (iv), we deduce that $\vartheta(\delta) \to 0$, as $\delta \to 0$ independently on $x \in Q$.
Now, to prove the operator A maps $C[0,1]$ into itself, let $t_1, t_2 \in [0,1]$, such that $|t_2 - t_1| < \delta$, then we have

$$|Ax(t_2) - Ax(t_1)| =$$

$$\left|p(t_2) + \int_0^{\varphi(t_2)} g_1\left(t_2, s, x(\omega(s))\right) d_s k_1(t_2,s) \int_0^1 g_2\left(t_2, s, x(\omega(s))\right) d_s k_2(t_2,s)\right.$$

$$\left. - p(t_1) + \int_0^{\varphi(t_1)} g_1\left(t_1, s, x(\omega(s))\right) d_s k_1(t_1,s) \int_0^1 g_2\left(t_1, s, x(\omega(s))\right) d_s k_2(t_1,s)\right|$$

$$\le |p(t_2) - p(t_1)|$$

$$+\Big|\int_0^{\varphi(t_2)} g_1\left(t_2,s,x(\omega(s))\right)d_s k_1(t_2,s)\int_0^1 g_2\left(t_2,s,x(\omega(s))\right)d_s k_2(t_2,s)$$

$$-\int_0^{\varphi(t_1)} g_1\left(t_1,s,x(\omega(s))\right)d_s k_1(t_1,s)\int_0^1 g_2\left(t_2,s,x(\omega(s))\right)d_s k_2(t_2,s)$$

$$+\int_0^{\varphi(t_1)} g_1\left(t_1,s,x(\omega(s))\right)d_s k_1(t_1,s)\int_0^1 g_2\left(t_2,s,x(\omega(s))\right)d_s k_2(t_2,s)$$

$$-\int_0^{\varphi(t_1)} g_1\left(t_1,s,x(\omega(s))\right)d_s k_1(t_1,s)\int_0^1 g_2\left(t_1,s,x(\omega(s))\right)d_s k_2(t_1,s)\Big|$$

$$\leq |p(t_2)-p(t_1)|$$

$$+\Big|\int_0^1 g_2\left(t_2,s,x(\omega(s))\right)d_s k_2(t_2,s)\Big[\int_0^{\varphi(t_2)} g_1\left(t_2,s,x(\omega(s))\right)d_s k_1(t_2,s)$$

$$-\int_0^{\varphi(t_1)} g_1\left(t_1,s,x(\omega(s))\right)d_s k_1(t_1,s)\Big]+\int_0^{\varphi(t_1)} g_1\left(t_1,s,x(\omega(s))\right)d_s k_1(t_1,s)$$

$$+\Big[\int_0^1 g_2\left(t_2,s,x(\omega(s))\right)d_s k_2(t_2,s)-\int_0^1 g_2\left(t_1,s,x(\omega(s))\right)d_s k_2(t_1,s)\Big]\Big|$$

$$\leq |p(t_2)-p(t_1)|+$$
$$\Big|\int_0^1 g_2\left(t_2,s,x(\omega(s))\right)d_s k_2(t_2,s)\Big[\int_0^{\varphi(t_1)} g_1\left(t_2,s,x(\omega(s))\right)d_s k_1(t_2,s)$$

$$+\int_{\varphi(t_1)}^{\varphi(t_2)} g_1\left(t_2,s,x(\omega(s))\right)d_s k_1(t_2,s)-\int_0^{\varphi(t_1)} g_1\left(t_1,s,x(\omega(s))\right)d_s k_1(t_1,s)\Big]$$

$$+\int_0^{\varphi(t_1)} g_2\left(t_1,s,x(\omega(s))\right)d_s k_2(t_1,s)\Big[\int_0^1 g_2\left(t_2,s,x(\omega(s))\right)d_s k_2(t_2,s)$$

$$-\int_0^1 g_2\left(t_2,s,x(\omega(s))\right)d_s k_2(t_1,s)+\int_0^1 g_2\left(t_2,s,x(\omega(s))\right)d_s k_2(t_1,s)$$

$$-\int_0^1 g_2\left(t_1,s,x(\omega(s))\right)d_s k_2(t_1,s)\Big]\Big|$$

$$\leq |p(t_2)-p(t_1)|$$

$$+\Big|\int_0^1 g_2\left(t_2,s,x(\omega(s))\right)d_s k_2(t_2,s)\Big[\int_0^{\varphi(t_1)} g_1\left(t_2,s,x(\omega(s))\right)d_s k_1(t_2,s)$$

$$+\int_{\varphi(t_1)}^{\varphi(t_2)} g_1\left(t_2,s,x(\omega(s))\right)d_s k_1(t_2,s)-\int_0^{\varphi(t_1)} g_1\left(t_1,s,x(\omega(s))\right)d_s k_1(t_1,s)$$

$$+\int_0^{\varphi(t_1)} g_1\left(t_2,s,x(\omega(s))\right)d_s k_1(t_1,s)-\int_0^{\varphi(t_1)} g_1\left(t_2,s,x(\omega(s))\right)d_s k_1(t_1,s)\Big]$$

$$+\int_0^{\varphi(t_1)} g_2\left(t_1,s,x(\omega(s))\right)d_s k_2(t_1,s)$$

$$.\ \Big[\int_0^1 g_2\left(t_2,s,x(\omega(s))\right)d_s[k_2(t_2,s)-k_2(t_1,s)]$$

$$+\int_0^1\left[g_2\left(t_2,s,x(\omega(s))\right)-g_2\left(t_1,s,x(\omega(s))\right)\right]d_sk_2(t_1,s)\Big]\Big|$$

$$\leq|p(t_2)-p(t_1)|\left|\int_0^1 f_2\left(t_2,s,x(\omega(s))\right)d_sg_2(t_2,s)\left[\int_{\varphi(t_1)}^{\varphi(t_2)} f_1\left(t_2,s,x(\omega(s))\right)d_s\,g_1(t_2,s)\right.\right.$$

$$+\int_0^{\varphi(t_1)} f_1\left(t_2,s,x(\omega(s))\right)d_s[g_1(t_2,s)-g_1(t_1,s)]$$

$$+\int_0^{\varphi(t_1)}[f_1\left(t_2,s,x(\omega(s))\right)-f_1\left(t_1,s,x(\omega(s))\right)]d_sg_1(t_1,s)\Big]$$

$$+\int_0^{\varphi(t_1)} g_2\left(t_1,s,x(\omega(s))\right)d_sk_2(t_1,s)$$

$$.\left[\int_0^1 g_2\left(t_2,s,x(\omega(s))\right)d_s[k_2(t_2,s)-k_2(t_1,s)]\right.$$

$$+\int_0^1\left[g_2\left(t_2,s,x(\omega(s))\right)-g_2\left(t_1,s,x(\omega(s))\right)\right]d_sk_2(t_1,s)\Big]\Big|$$

$$\leq\left|p(t_2)-p(t_1)+(M+br)[g_2(t_2,1)-g_2(t_2,0)]\right|$$

$$.\left[(M+br)\bigvee_{\varphi(t_1)}^{\varphi(t_2)} g_1(t_2,s)+(M+br)N(\epsilon)+\vartheta(\epsilon)\sup_{t\in I}\left(\bigvee_0^{\varphi(t_1)} g_1(t_1,s)\right)\right]$$

$$+(M+br)\sup_{t\in I}\left(\bigvee_0^{\varphi(t_1)} g_2(t_1,s)\right)\left[(M+br)\bigvee_{s=0}^{z}[g_2(t_2,s)-g_2(t_1,s)]\right.$$

$$+\vartheta(\epsilon)(g_2(t_1,1)-g_2(t_1,0)],$$

where

$$N(\epsilon)=sup\bigvee_{s=0}^{t_1}(g_1(t_2,s)-g_1(t_1,s)):t_1,t_2\in I,t_1<t_2,t_2-t_1\leq\omega$$

the above inequality means that $Ax:C[0,1]\to C[0,1]$.
Then AQ is compact.
Now we prove that the operator A is continuous.
Let $\{x_n\}\subset Q$, and $\{x_n\}\to x, Q\subseteq R$ then

$$Ax_n(t)=p(t)+\int_0^{\varphi(t)} g_1(t,s,x_n(\omega(s)))d_sk_1(t,s)\int_0^1 g_2(t,s,x_n(\omega(s)))d_sk_2(t,s)$$

$$\lim_{n\to\infty}Ax_n(t)=p(t)+\lim_{n\to\infty}\int_0^{\varphi(t)} g_1(t,s,x_n(\omega(s)))d_sk_1(t,s)\int_0^1 g_2(t,s,x_n(\omega(s)))d_sk_2(t,s).$$

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Applying Lebesgue dominated convergence theorem [14], then

$$= \boldsymbol{p}(\boldsymbol{t}) + \int_0^{\varphi(t)} \boldsymbol{g_1}(\boldsymbol{t}, \boldsymbol{s}, \lim_{n\to\infty} \boldsymbol{x_n}(\boldsymbol{\omega}(\boldsymbol{s})))\boldsymbol{d_s k_1}(\boldsymbol{t}, \boldsymbol{s}) \int_0^1 \boldsymbol{g_2}(\boldsymbol{t}, \boldsymbol{s}, \lim_{n\to\infty} \boldsymbol{x_n}(\boldsymbol{\omega}(\boldsymbol{s})))\boldsymbol{d_s k_2}(\boldsymbol{t}, \boldsymbol{s})$$

$$= \boldsymbol{p}(\boldsymbol{t}) + \int_0^{\varphi(t)} \boldsymbol{g_1}(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{x_0}(\boldsymbol{\omega}(\boldsymbol{s})))\boldsymbol{d_s k_1}(\boldsymbol{t}, \boldsymbol{s}) \int_0^1 \boldsymbol{g_2}(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{x_0}(\boldsymbol{\omega}(\boldsymbol{s})))\boldsymbol{d_s k_2}(\boldsymbol{t}, \boldsymbol{s}) = \boldsymbol{A x_0}(\boldsymbol{t})$$

which means that the operator $\boldsymbol{A}$ is continuous.
Since all conditions of Schauder fixed point theorem [14] (see also [2, 3, 6, 13]) are satisfied, the operator $\boldsymbol{A}$ has at least one fixed point $\boldsymbol{x} \in \boldsymbol{Q}$, and the integral equation (1) has at least one solution $\boldsymbol{x} \in \boldsymbol{C}[\boldsymbol{0}, \boldsymbol{1}]$. This completes the proof.

3 Uniqueness of the solution

This section deals with the uniqueness of the solution of the functional integral equation (1), we replace the assumption (iv) by
(iv)* $\boldsymbol{g_i}: [\boldsymbol{0}, \boldsymbol{1}] \times [\boldsymbol{0}, \boldsymbol{1}] \times \boldsymbol{R} \to \boldsymbol{R}, \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}$ are continuous and satisfy the Lipschitz condition

$$|\boldsymbol{g_1}(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{x}) - \boldsymbol{g_1}(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{y})| \le \boldsymbol{b_1}|\boldsymbol{x} - \boldsymbol{y}|, \qquad \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{Q},$$

$$|\boldsymbol{g_2}(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{x}) - \boldsymbol{g_2}(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{y})| \le \boldsymbol{b_2}|\boldsymbol{x} - \boldsymbol{y}|, \quad \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{Q}$$

and $\boldsymbol{b} = \boldsymbol{max}\{\boldsymbol{b_1}, \boldsymbol{b_2}\}$.

From assumption (iv)* we have consecutively

$$|\boldsymbol{g_1}(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{x}(\boldsymbol{\omega}(\boldsymbol{s})))| - |\boldsymbol{g_1}(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{0})| \le \left|\boldsymbol{g_1}\left(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{x}(\boldsymbol{\omega}(\boldsymbol{s}))\right) - \boldsymbol{g_1}(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{0})\right| \le \boldsymbol{b}|\boldsymbol{x}|,$$

$$|\boldsymbol{g_1}(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{x}(\boldsymbol{\omega}(\boldsymbol{s})))| \le \boldsymbol{b}|\boldsymbol{x}| + |\boldsymbol{g_1}(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{0})|.$$

Hence,

$$|\boldsymbol{g_1}(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{x}(\boldsymbol{\omega}(\boldsymbol{s})))| \le \boldsymbol{b}|\boldsymbol{x}| + \boldsymbol{M}, \qquad \boldsymbol{M} = \sup_{\boldsymbol{t}} \{\boldsymbol{g_1}(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{0}) : \boldsymbol{t}, \boldsymbol{s} \in [\boldsymbol{0}, \boldsymbol{1}], \qquad \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}\}.$$

Similarly,

$$|\boldsymbol{g_2}(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{x}(\boldsymbol{\omega}(\boldsymbol{s})))| \le \boldsymbol{b}|\boldsymbol{x}| + \boldsymbol{M}.$$

For the uniqueness of the solution of the functional integral equation (1) we have following theorem.

Theorem 2. Let assumptions (i)-(ii)-(iii)-(iv)*-(v)-(vi)-(vii)-(viii)-(x)-(ix) be satisfied, if $2(M+br)W\mu b<1$, then the solution $x\in C[0,1]$ of the functional equation (1) is unique.

Proof. Let x_1, x_2 be two solutions of the integral equation (1), then

$$|Ax_1 - Ax_2| = |Ax_1(t) - Ax_2(t)|$$

$$= \Big| p(t) + \int_0^{\varphi(t)} g_1\big(t,s,x_1(\omega(s))\big)\, d_s k_1(t,s) \int_0^1 g_2\big(t,s,x_1(\omega(s))\big)\, d_s k_2(t,s)$$

$$- p(t) - \int_0^{\varphi(t)} g_1\big(t,s,x_2(\omega(s))\big)\, d_s k_1(t,s) \int_0^1 g_2\big(t,s,x_2(\omega(s))\big)\, d_s k_2(t,s)\Big|$$

$$= \Big| \int_0^{\varphi(t)} g_1\big(t,s,x_1(\omega(s))\big)\, d_s k_1(t,s) \int_0^1 g_2\big(t,s,x_1(\omega(s))\big)\, d_s k_2(t,s)$$

$$- \int_0^{\varphi(t)} g_1\big(t,s,x_2(\omega(s))\big)\, d_s k_1(t,s) \int_0^1 g_2\big(t,s,x_2(\omega(s))\big)\, d_s k_2(t,s)\Big|$$

$$= \Big| \int_0^{\varphi(t)} g_1\big(t,s,x_1(\omega(s))\big)\, d_s k_1(t,s) \int_0^1 g_2\big(t,s,x_1(\omega(s))\big)\, d_s k_2(t,s)$$

$$- \int_0^{\varphi(t)} g_1\big(t,s,x_2(\omega(s))\big)\, d_s k_1(t,s) \int_0^1 g_2\big(t,s,x_2(\omega(s))\big)\, d_s k_2(t,s)$$

$$+ \int_0^{\varphi(t)} g_1\big(t,s,x_1(\omega(s))\big)\, d_s k_1(t,s) \int_0^1 g_2\big(t,s,x_2(\omega(s))\big)\, d_s k_2(t,s)$$

$$- \int_0^{\varphi(t)} g_1\big(t,s,x_2(\omega(s))\big)\, d_s k_1(t,s) \int_0^1 g_2\big(t,s,x_2(\omega(s))\big)\, d_s k_2(t,s)\Big|$$

$$\le \Big| \int_0^{\varphi(t)} g_1\big(t,s,x_1(\omega(s))\big)\, d_s k_1(t,s)$$

$$.\Big[\int_0^1 \Big(g_2\big(t,s,x_1(\omega(s))\big) - g_2\big(t,s,x_2(\omega(s))\big)\Big)\, d_s k_2(t,s)\Big]\Big|$$

$$+ \Big| \int_0^1 g_2\big(t,s,x_2(\omega(s))\big)\, d_s k_2(t,s)$$

$$.\Big[\int_0^{\varphi(t)} \Big(g_1\big(t,s,x_1(\omega(s))\big) - g_1\big(t,s,x_2(\omega(s))\big)\Big)\, d_s k_2(t,s)\Big]\Big|$$

$$\le (M+br)W\mu b|x_1 - x_2| + (M+br)W\mu b|x_1 - x_2|,$$

$$|Ax_1 - Ax_2| \le 2(M+br)W\mu b\|x_1 - x_2\|.$$

This proves that the map $A: C[0,1] \to C[0,1]$ is a contraction. Then by Banach fixed point theorem the solution of the functional integral equation (1) is unique.

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4 Continuous dependence of the solution

To study the continuous dependence of the unique solution $x \in C[0,1]$ of the functional integral equation (1) on the delay function and the functions k_1, k_2.

4.1 Continuous dependence on the delay function $\varphi(t)$

Defintion1. The solution of the functional integral equation (1), depends continuously on the delay function $\varphi(t)$ if $\forall \epsilon > 0, \exists\, \delta > 0,$ such that

$$|\varphi(t) - \varphi^*(t)| \le \delta \Rightarrow \|x - x^*\| \le \epsilon.$$

Theorem 3. Let the assumptions of Theorem 2 be satisfied. Then the solution of the functional integral equation (1) depends continuously on the delay function $\varphi(t)$.

Proof. Let $\delta > 0$ be given such that $|\varphi(t) - \varphi^*(t)| \le \delta,\ \forall\, t > 0$, then

$$|x(t) - x^*(t)| = \Big| p(t) + \int_0^{\varphi(t)} g_1\big(t, s, x(\omega(s))\big)\, d_s k_1(t,s) \int_0^1 g_2\big(t, s, x(\omega(s))\big)\, d_s k_2(t,s)$$

$$- p(t) + \int_0^{\varphi^*(t)} g_1\big(t, s, x^*(\omega(s))\big)\, d_s k_1(t,s) \int_0^1 g_2\big(t, s, x^*(\omega(s))\big)\, d_s k_2(t,s)\Big|$$

$$\le \Big| \int_0^{\varphi(t)} g_1\big(t, s, x(\omega(s))\big)\, d_s k_1(t,s) \int_0^1 g_2\big(t, s, x(\omega(s))\big)\, d_s k_2(t,s)$$

$$- \int_0^{\varphi^*(t)} g_1\big(t, s, x^*(\omega(s))\big)\, d_s k_1(t,s) \int_0^1 g_2\big(t, s, x^*(\omega(s))\big)\, d_s k_2(t,s)$$

$$+ \int_0^{\varphi^*(t)} g_1\big(t, s, x^*(\omega(s))\big)\, d_s k_1(t,s) \int_0^1 g_2\big(t, s, x(\omega(s))\big)\, d_s k_2(t,s)$$

$$- \int_0^{\varphi^*(t)} g_1\big(t, s, x^*(\omega(s))\big)\, d_s k_1(t,s) \int_0^1 g_2\big(t, s, x(\omega(s))\big)\, d_s k_2(t,s)\Big|$$

$$\le \Big| \int_0^1 g_2\big(t, s, x(\omega(s))\big)\, d_s k_2(t,s)$$

$$.\ \Big[\int_0^{\varphi(t)} g_1\big(t, s, x(\omega(s))\big)\, d_s k_1(t,s) - \int_0^{\varphi^*(t)} g_1\big(t, s, x^*(\omega(s))\big)\, d_s k_1(t,s) \Big] \Big|$$

$$+ \Big| \int_0^{\varphi^*(t)} g_1\big(t, s, x^*(\omega(s))\big)\, d_s k_1(t,s)$$

$$. \left[\int_0^1 g_2\left(t,s,x(\omega(s))\right) d_s k_2(t,s) - \int_0^1 g_2\left(t,s,x^*(\omega(s))\right) d_s k_2(t,s)\right]\Big|$$

$$\le \Big|\int_0^1 g_2\left(t,s,x(\omega(s))\right) d_s k_2(t,s)\left[\int_o^{\varphi(t)} g_1\left(t,s,x(\omega(s))\right) d_s k_1(t,s)\right.$$

$$-\int_o^{\varphi(t)} g_1\left(t,s,x^*(\omega(s))\right) d_s k_1(t,s) + \int_o^{\varphi(t)} g_1\left(t,s,x^*(\omega(s))\right) d_s k_1(t,s)$$

$$\left.-\int_o^{\varphi^*(t)} g_1\left(t,s,x^*(\omega(s))\right) d_s k_1(t,s)\right]$$

$$+\int_0^{\varphi^*(t)} g_1\left(t,s,x^*(\omega(s))\right) d_s k_1(t,s)$$

$$. \left[\int_0^1 g_2\left(t,s,x(\omega(s))\right) d_s k_2(t,s) - \int_0^1 g_2\left(t,s,x^*(\omega(s))\right) d_s k_2(t,s)\right]\Big|$$

$$\le \int_0^1 \left|g_2\left(t,s,x(\omega(s))\right)\right| d_s k_2(t,s)\left[\int_o^{\varphi(t)} b|x(s)-x^*(s)| d_s k_1(t,s)\right.$$

$$\left.+\int_{\varphi^*(t)}^{\varphi(t)} \left|g_1\left(t,s,x^*(\omega(s))\right)\right| d_s k_1(t,s)\right]$$

$$+\int_0^{\varphi^*(t)} \left|g_1\left(t,s,x^*(\omega(s))\right)\right| d_s k_1(t,s)[b|x(s)-x^*(s)| d_s k_2(t,s)]$$

$$\le (M+br)[k_2(t,1)-k_2(t,0)]\left\{b|x-x^*| \sup_{t\in I}\left(\bigvee_{s=0}^{\varphi(t)} k_1(t,s)\right)\right.$$

$$\left.+(M+br)[k_1(t,\varphi(t)) - k_1(t,\varphi^*(t))]\right\}$$

$$+(M+br)\sup_{t\in I}\left(\bigvee_{s=0}^{\varphi^*(t)} k_1(t,s)\right)\{b|x-x^*|(k_2(t,1)-k_2(t,0))\}$$

$$\le (M+br)\mu\{b\|x-x^*\|W + (M+br)[\,k_1(t,\varphi(t)) - k_1(t,\varphi^*(t))]$$

$$+(M+br)Wb\|x-x^*\|\mu,$$

$$\|x-x^*\| \le \frac{(M+br)^2\mu[\,k_1(t,\varphi(t))-k_1(t,\varphi^*(t))]}{1-2(M+br)W\mu b}.$$

Using the continuity of k_1 we have

$$|\varphi(t)-\varphi^*(t)| \le \delta \Rightarrow \|x-x^*\| \le \epsilon_1.$$

then

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$$\leq \frac{(M+br)^2\mu[\,k_1(t,\varphi(t))-k_1(t,\varphi^*(t))]}{1-2(M+br)W\mu b}=\epsilon.$$

This completes the proof.

4.2 Continuous dependence on the functions k_1

Definition 2. The solution of the quadratic functional integral equation (1), depends continuously on the functions $k_i(t,s),\ i=1,2$ if $\forall\epsilon>0, \exists\ \delta>0$, such that

$$|k_i(t,s)-k_i^*(t,s)|\leq\delta\Longrightarrow\|x-x^*\|\leq\epsilon.$$

Theorem 4. Let the assumptions of Theorem 2 be satisfied, then the solution of the delay quadratic functional integral equation (1) depends continuously on the functions k_1, k_2.

Proof. Let $\delta>0$ be given such that $|k_i(t,s)-k_i^*(t,s)|\leq\delta,\ \forall\ t>0$, then

$$|x(t)-x^*(t\,)| = \left|p(t)+\int_0^{\varphi(t)} g_1\left(t,s,x(\omega(s))\right)d_sk_1(t,s)\int_0^1 g_2\left(t,s,x(\omega(s))\right)d_sk_2(t,s)\right.$$

$$-\,p(t)+$$

$$\left.\int_0^{\varphi(t)} g_1\left(t,s,x^*(\omega(s))\right)d_sk_1^*(t,s)\int_0^1 g_2\left(t,s,x^*(\omega(s))\right)d_sk_2^*(t,s)\right|$$

$$\leq\left|\int_0^{\varphi(t)} g_1\left(t,s,x(\omega(s))\right)d_sk_1(t,s)\int_0^1 g_2\left(t,s,x(\omega(s))\right)d_sk_2(t,s)\right.$$

$$-\int_0^{\varphi(t)} g_1\left(t,s,x(\omega(s))\right)d_sk_1(t,s)\int_0^1 g_2\left(t,s,x^*(\omega(s))\right)d_sk_2^*(t,s)$$

$$+\int_0^{\varphi(t)} g_1\left(t,s,x(\omega(s))\right)d_sk_1(t,s)\int_0^1 g_2\left(t,s,x^*(\omega(s))\right)d_sk_2^*(t,s)$$

$$\left.-\int_0^{\varphi(t)} g_1\left(t,s,x^*(\omega(s))\right)d_sk_1^*(t,s)\int_0^1 g_2\left(t,s,x^*(\omega(s))\right)d_sk_2^*(t,s)\right|$$

$$\leq\left|\int_0^{\varphi(t)} g_1\left(t,s,x(\omega(s))\right)d_sk_1(t,s)\right.$$

$$\left.\cdot\left[\int_o^1 g_2\left(t,s,x(\omega(s))\right)d_sk_2(t,s)-\int_o^1 g_2\left(t,s,x^*(\omega(s))\right)d_sk_2^*(t,s)\right]\right|$$

$$+\left|\int_0^1 g_2\left(t,s,x^*(\omega(s))\right)d_sk_2^*(t,s)\right.$$

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$$. \left[\int_0^{\varphi(t)} g_1\left(t,s,x(\omega(s))\right) d_s k_1(t,s) - \int_0^{\varphi(t)} g_1\left(t,s,x^*(\omega(s))\right) d_s k_1^*(t,s)\right]\Big|$$

$$\leq \left|\int_0^{\varphi(t)} g_1\left(t,s,x(\omega(s))\right) d_s k_1(t,s)\left[\int_o^1 g_2\left(t,s,x(\omega(s))\right) d_s k_2(t,s)\right.\right.$$

$$-\int_o^1 g_2\left(t,s,x(\omega(s))\right) d_s k_2^*(t,s) + \int_o^1 g_2\left(t,s,x(\omega(s))\right) d_s k_2^*(t,s)$$

$$\left.\left.-\int_o^1 g_1\left(t,s,x^*(\omega(s))\right) d_s k_2^*(t,s)\right]\right|$$

$$+\left|\int_0^1 g_2\left(t,s,x^*(\omega(s))\right) d_s k_2^*(t,s)\left[\int_0^{\varphi(t)} g_1\left(t,s,x(\omega(s))\right) d_s k_1(t,s)\right.\right.$$

$$-\int_0^{\varphi(t)} g_1\left(t,s,x(\omega(s))\right) d_s k_1^*(t,s) + \int_0^{\varphi(t)} g_1\left(t,s,x(\omega(s))\right) d_s k_1^*(t,s)$$

$$\left.\left.-\int_0^{\varphi(t)} g_1\left(t,s,x^*(\omega(s))\right) d_s k_1^*(t,s)\right]\right|$$

$$\leq \left|\int_0^{\varphi(t)} g_1\left(t,s,x(\omega(s))\right) d_s k_1(t,s)\left[\int_o^1 g_2\left(t,s,x(\omega(s))\right) d_s[k_2(t,s) - k_2^*(t,s)]\right.\right.$$

$$\left.\left.+\int_o^1\left[g_2\left(t,s,x(\omega(s))\right) - g_2\left(t,s,x^*(\omega(s))\right)\right] d_s k_2^*(t,s)\right]\right|$$

$$+\left|\int_0^1 g_2\left(t,s,x^*(\omega(s))\right) d_s k_2^*(t,s)\left[\int_0^{\varphi(t)} g_1\left(t,s,x(\omega(s))\right) d_s[k_1(t,s) - k_1^*(t,s)]\right.\right.$$

$$\left.\left.+\int_0^{\varphi(t)} g_1\left(t,s,x(\omega(s))\right) - g_1\left(t,s,x^*(\omega(s))\right) d_s k_1(t,s)\right]\right|$$

$$\leq (M+br)\sup_{t\in I}\left(\vee_{s=0}^{\varphi(t)} k_1(t,s)\right)\{(M+br)|[k_2(t,1) - k_2^*(t,1)]$$

$$-[k_2(t,0) - k_2^*(t,0)]| + b|x - x^*|[k_2^*(t,1) - k_2^*(t,0)\}$$

$$+(M+br)[k_1^*(t,1) - 1(t,1)]\{(M+br)[k_1(t,\varphi(t)) - k_1^*(t,\varphi(t))]$$

$$-[k_1(t,0) - k_1^*(t,\varphi(t)) + b|x - x^*|\sup_{t\in I}\left(\vee_{s=0}^{\varphi(t)} k_1(t,s)\right)\}$$

$$\leq 2(M+br)^2 W\delta + (M+br)W\mu b\|x - x^*\| + 2(M+br)^2\mu\delta + (M+br)Wb\mu\|x - x^*\|,$$

then

$$\|x - x^*\|[1 - 2W\mu b(M+br)] \leq 4(M+br)^2\delta[W+\mu]$$

$$\|x - x^*\| \leq \frac{4(M+br)^2\delta[W+\mu]}{[1 - 2W\mu b(M+br)]} = \epsilon.$$

This completes the proof.

5. Example

Let $g_i(t,s,x(\omega(s))) = k_i(t,s)|x(s)|$ and let functions $k_i(t,s)$ be given by

$$k_i(t,s) = \begin{cases} t\ln\frac{t+s}{t}, & t \in (0,T] \\ 0, & t = 0, \end{cases}$$

Then

$$dk_i(t,s) = \frac{t}{t+s}ds$$

and the assumptions (iv) – (vii) are satisfied (see [11]). Then our result can be applied to the delay Volterra quadratic integral equation of Chandresekhar's type (2) and the unique solution of (2) depends continuous on the delay functions $\varphi(t)$.

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